TWO SLITS IN A STRIP OF FINITE THICKNESS

PMM Vol. 34, №2, 1970, pp. 366-369

B. I. SMETANIN
(Rostov-on-Don)

(Received May 20, 1969)

We consider the following two plane mixed problems of the theory of elasticity. The problem of stretching a strip weakened by two equal length longitudinal slits, and the problem of longitudinal splitting of a strip by means of a wedge of finite length (*). In the second problem, slits appear on each side of the wedge. Both, the slits and the wedge are symmetrical with respect to the edges of the strip.

The method of solution employed is analogous to that developed in [2]. The formulas obtained define the form of the slit surface and the normal stress intensity coefficient, and represent asymptotic expansions of the exact solutions in negative powers of the parameter $\lambda = h / b$ characterizing the relative thickness of the strip.

1. The problem of stretching a strip weakened by two slits. Two slits of equal length $b - a$ (Fig. 1) are present in an elastic infinite strip of thickness $2h$. The slits are symmetrical with respect to the strip edges and the surface of the slits is load-free. The edges of the strip are either displaced outwards by a given distance δ, or are acted upon by an uniformly distributed tensile load of intensity p. We require to find the form of the slit surface $\gamma(x)$ and the normal stress intensity coefficient N, the stress appearing outside the slits on the line produced from the slits. The boundary conditions of the problem are

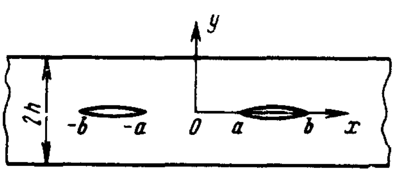

Fig. 1

for $y = 0$

$$\sigma_y = 0 \ (a \leqslant |x| \leqslant b), \quad u_y = 0 \ (|x| \leqslant a, \ \ b \leqslant |x|), \quad \tau_{xy} = 0 \ (0 \leqslant |x|) \tag{1.1}$$

for $y = \pm h, \ \ 0 \leqslant |x|$

$$(1) \ u_y = \pm\delta, \ \tau_{xy} = 0, \quad (2) \ u_y = \pm\delta, \ u_x = 0, \quad (3) \ \sigma_y = p, \ \tau_{xy} = 0 \tag{1.2}$$

The plus and minus signs correspond to the upper and lower edge of the strip. The problem can be reduced to an auxiliary problem with the following boundary conditions:

for $y = 0$

$$\sigma_y = -q \ (a \leqslant |x| \leqslant b), \quad u_y = 0 \quad (|x| \leqslant a, \ b \leqslant |x|), \quad \tau_{xy} = 0 \ (0 \leqslant |x|) \tag{1.3}$$

for $y = \pm h, \ 0 \leqslant |x|$

$$(1) \ u_y = 0, \ \tau_{xy} = 0, \quad (2) \ u_y = 0, \ u_x = 0, \quad (3) \ \sigma_y = 0, \ \tau_{xy} = 0 \tag{1.4}$$

For the conditions (1) and (2)

$$q = 2G\delta h^{-1} (1 - \nu)(1 - 2\nu)^{-1}$$

where G is the shear modulus and ν is the Poisson's ratio. For the condition (3) $q = p$. The problem with the boundary conditions (1.1) and (1.2) cna be solved by superimposing the stress and strain fields on the solution of the auxiliary problem. These fields

---

*) Contact problems for a strip with two equal regions of contact were dealt with in [1].

correspond to the displacement vector whose components are

$$u_x = 0, \qquad u_y = h^{-1}\delta y$$

for the conditions (1) and (2) on the edges of the strip and

$$u_x = 0, \qquad u_y = 0.5pG^{-1}(1-\nu)^{-1}(1-2\nu)\, y$$

for the condition (3).

The function $\gamma(x)$ and the quantity N are the same for both, the initial and the auxiliary problems. By symmetry, it is obvously sufficient to consider the region $0 \leqslant y \leqslant h$, $-\infty < x < \infty$.

Methods of operational calculus are employed [3] to reduce the problem with the boundary conditions (1.3) and (1.4) to that of obtaining the function $\gamma'(x)$ from the following integral equation:

$$\left(\int_{-b}^{-a} + \int_{a}^{b}\right)\gamma'(\xi)\, Q\left(\frac{\xi - x}{h}\right) d\xi = -\pi h q \frac{(1-\nu)}{G} \qquad (a \leqslant |x| \leqslant b) \tag{1.5}$$

$$Q(t) = \int_0^\infty L(u) \sin(ut)\, du \tag{1.6}$$

For conditions (1) to (3) the corresponding expressions for $L(u)$ are

(1) $L(u) = \dfrac{\mathrm{sh}\, 2u + 2u}{\mathrm{ch}\, 2u - 1}$

(2) $L(u) = \dfrac{\varkappa\, \mathrm{ch}\, 2u + 2u^2 + 0.5(1+\varkappa^2)}{\varkappa\, \mathrm{sh}\, 2u - 2u} \qquad (\varkappa = 3 - 4\nu)$

(3) $L(u) = 2\dfrac{\mathrm{sh}^2 u - u^2}{\mathrm{sh}\, 2u + 2u}, \quad L(u) \to 1 + O(e^{-2u})$ when $u \to \infty$

Let us represent the kernel $Q(t)$ of the integral equation (1.5) in the form

$$Q(t) = \frac{1}{t} + \sum_{i=0}^{\infty} c_i t^{2i+1} \tag{1.7}$$

The constants c_i are given by

$$c_i = \frac{(-1)^i}{(2i+1)!}\int_0^\infty [L(u) - 1]\, u^{2i+1} du \qquad (i = 0, 1, \ldots) \tag{1.8}$$

and assume the following values for the conditions (1), (2) and (3):

(1) $c_i = \dfrac{2i+3}{(2i+2)!}\pi^{2i+2} B_{2i+2} \qquad (i = 0, 1, \ldots)$

(2) $(\nu = 0.3) \quad c_0 = 3.48, \quad c_1 = -1.25, \quad c_2 = 0.518$

(3) $c_0 = -2.35, \quad c_1 = 1.69, \quad c_2 = -0.844$

Here B_{2m} are the Bernoulli numbers [4].

Inserting (1.7) into (1.5) and utilizing the fact that $\gamma'(-x) = -\gamma'(x)$, we obtain after simple transformations

$$\int_a^b \frac{\xi \gamma'(\xi)\, d\xi}{\xi^2 - x^2} = -\frac{\pi q (1-\nu)}{2G} - \frac{1}{2}\sum_{i=0}^{\infty} \frac{c_i}{h^{2i+2}} \int_a^b \gamma'(\xi) \left[(\xi + x)^{2i+1} + (\xi - x)^{2i+1}\right] d\xi \tag{1.9}$$

Application of the inversion formula to (1.9) yields an integral equation of the second kind in $\gamma'(x)$

$$\gamma'(x)=\frac{2}{\pi R(x)}\left\{P_1+\frac{q\pi(1-\nu)}{4G}(a^2+b^2-2x^2)+\right.$$
$$\left.+\frac{1}{\pi}\sum_{i=0}^{\infty}\frac{c_i b^{-2i-2}}{\lambda^{2i+2}}\int_a^b\frac{tR(t)\,dt}{t^2-x^2}\int_a^b\gamma'(\xi)\,[(\xi+t)^{2i+1}+(\xi-t)^{2i+1}]\,d\xi\right. \qquad (1.10)$$
$$R(x)=\sqrt{(x^2-a^2)(b^2-x^2)}$$

where P_1 is a constant to be defined. Seeking the solution of (1.10) in the form of a power series in λ^{-2} we obtain

$$\gamma'(x)=(1-\nu)\,G^{-1}R^{-1}(x)\,[P_1\Phi_1(x)+\Phi_2(x)] \qquad (a\leqslant x\leqslant b) \qquad (1.11)$$
$$\Phi_1(x)=G\pi^{-1}(1-\nu)^{-1}\{2+(1+\varepsilon^2-2x^2/b^2)\,c_0\lambda^{-2}+[{}^1/_4(5-2\varepsilon^2+5\varepsilon^4)+$$
$$+2(1+\varepsilon^2)\,x^2/b^2-6x^4/b^4]\,c_1\lambda^{-4}+O(\lambda^{-6})$$
$$\Phi_2(x)={}^1/_2 qb^2\beta\,(1+\varepsilon^2-2x^2/b^2)+O(\lambda^{-6}) \qquad (1.12)$$
$$\varepsilon=a/b,\qquad \beta=1-{}^1/_8\,c_1k^4\lambda^{-4},\quad k=\sqrt{1-\varepsilon^2}$$

Integrating (1.10) with respect to x and recalling that $\gamma(a)=0$, we obtain

$$\gamma(x)=P_1(\pi b)^{-1}\Omega(x)+{}^1/_2\,qb\beta G^{-1}(1-\nu)\,\chi(x)+O(\lambda^{-6}) \qquad (a\leqslant x\leqslant b) \qquad (1.13)$$
$$\Omega(x)=2F(\omega,k)+\chi(x)\,c_0\lambda^{-2}+[(1+\varepsilon^2)\chi(x)+{}^1/_4k^4F(\omega,k)+2xb^{-3}R(x)]\,c_1\lambda^{-}$$
$$\chi(x)=(1+\varepsilon_2)\,F(\omega,k)-2E(\omega,k)+2(bx)^{-1}R(x),\quad \omega=\arcsin\left(k^{-1}\sqrt{1-(a/x)^{2}}\right)$$

where $F(\omega,k)$ and $E(\omega,k)$ are elliptic integrals of the first and second kind respectively. The constant P_1 is found from the condition

$$\gamma(b)=0 \qquad (1.14)$$

Setting $\gamma(x)$ from (1.13) into (1.14) we obtain

$$P_1=-{}^1/_2 q\pi b^2\beta\,(1-\nu)\,G^{-1}\chi(b)\Omega^{-1}(b) \qquad (1.15)$$

The normal stress intensity coefficient N is defined for the points $x=a$ and $x=b$ by

$$N_a=\lim_{x\to a-0}\sqrt{a-x}\,\sigma_y(x,0)=\lim_{x\to a+0}\sqrt{x-a}\,\frac{G}{1-\nu}\gamma'(x)$$
$$N_b=\lim_{x\to b+0}\sqrt{x-b}\,\sigma_y(x,0)=-\lim_{x\to b-0}\sqrt{b-x}\,\frac{G}{1-\nu}\gamma'(x) \qquad (1.16)$$

Insertion of $\gamma'(x)$ from (1.11) into the right-hand sides of (1.16) yields

$$N_a=\frac{P_1\Phi_1(a)+\Phi_2(a)}{bk\sqrt{2a}},\qquad N_b=-\frac{P_1\Phi_1(b)+\Phi_2(b)}{bk\sqrt{2b}} \qquad (1.17)$$

Absolute convergence of the series (1.7) for $t<2$ can be shown using the properties of the function $L(u)$. Hence, the results (1.13) and (1.17) obtained are valid for $1<\lambda<\infty$. In practice, the relations obtained can be employed rationally within the range $2\leqslant\lambda<\infty$. When $\lambda\to\infty$, we have from (1.12) and (1.15)

$$P_1\Phi_1(a)+\Phi_2(a)=qb^2\,[E(k)/K(k)-\varepsilon^2],$$
$$P_1\Phi_1(b)+\Phi_2(b)=-qb^2\,[1-E(k)/K(k)] \qquad (1.18)$$

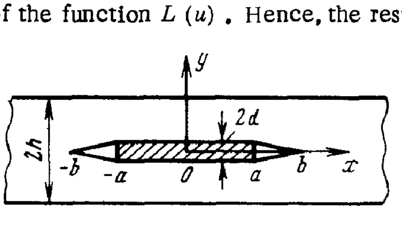

Fig. 2

Here $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind respectively. From (1.17) and (1.18) it follows that the quantities N_a and N_b coincide, as $\lambda\to\infty$, with the corresponding quantities obtained in the course of solution of the problem of two slits in a plane [6]. When $\lambda\to\infty$, the function $\gamma(x)$ coincides with the analogous function obtained in [7] during the investigation of a problem of a slit in a semiplane whose boundary is clamped, although free to slide. The case investigated in [7] also corresponds to the problem of two collinear slits of equal length on a plane.

2. The problem of splitting a strip. Let an elastic infinite strip of thickness $2h$ be split by means of a rigid, smooth wedge of length $2a$. The wedge is situated symmetrically with respect to the edges of the strip, and the wedge thickness is $2d$. A gap of length $b - a$ (Fig. 2) appears on each side of the wedge. The strip edges are displaced either outwards or inwards by a given distance δ, or are acted upon by an uniformly distributed tensile or compressive load of intensity p. We require to find the form of the slit surfaces $\gamma(x)$ and the normal stress intensity coefficient N at the points $y = 0$, $x = \pm(b + 0)$. The boundary conditions of the problem have the form

$$\text{for} \quad y = 0 \quad u_y = \pm d \ (|x| \leqslant a), \ u_y = 0 \ (b \leqslant |x|)$$
$$\sigma_y = 0 \ (a \leqslant |x| \leqslant b), \quad \tau_{xy} = 0 \quad (0 \leqslant |x|)$$

One of the three conditions given in (1.2) holds at the strip edges for $y = \pm h$, $0 \leqslant |x|$. The problem can be reduced to an auxiliary problem with the boundary conditions

$$\text{for} \quad y = 0 \quad u_y = \pm d \ (|x| \leqslant a), \quad u_y = 0 \quad (b \leqslant |x|)$$
$$\sigma_y = -q \quad (a \leqslant |x| \leqslant b), \quad \tau_{xy} = 0 \quad (0 \leqslant |x|) \tag{2.1}$$

using the method of superposition. Here one of the three conditions given in (1.4) holds for $y = \pm h$; we have, as before, $q = 2G\delta h^{-1} \times (1 - \nu)(1 - 2\nu)^{-1}$ for conditions (1) and (2), and $q = p$ for condition (3).

Methods of operational calculus can be used to reduce the problem with the boundary conditions (2.1) and (1.4) to that of finding the function $\gamma'(x)$ from an integral equation of the form of (1.5). Consequently, the expression defining the function $\gamma'(x)$ for the problem considered, should have the form

$$\gamma'(x) = (1 - \nu) G^{-1} R^{-1}(x) \left[P_2 \Phi_1(x) + \Phi_2(x)\right] \qquad (a \leqslant x \leqslant b) \tag{2.2}$$

where $\Phi_1(x)$ and $\Phi_2(x)$ are given by (1.12) and P_2 is a constant to be determined. Unlike in the problem in Sect. 1, the function $\gamma(x)$ in the present problem must satisfy the conditions

$$\gamma(a) = d, \quad \gamma(b) = 0 \tag{2.3}$$

hence we shall use the relation

$$\gamma(x) = d + \int_a^x \gamma'(\xi)\, d\xi \qquad (a \leqslant x \leqslant b) \tag{2.4}$$

where $\gamma'(\xi)$ has the form of (2.2), to determine $\gamma(x)$. Performing the necessary computations according to (2.4), we obtain

$$\gamma(x) = d + P_2 (\pi b)^{-1} \Omega(x) + {}^1/_2 q b \beta G^{-1} (1 - \nu) \chi(x) + O(\lambda^{-6}) \tag{2.5}$$

The constant P_2 obtained from (2.5) and the second relation of (2.3), is

$$P_2 = -\pi b \Omega^{-1}(b) \left[d + {}^1/_2 q \beta G^{-1} (1 - \nu) \chi(b)\right] + O(\lambda^{-6}) \tag{2.6}$$

Inserting $\gamma'(x)$ given by (2.2) into the second relation of (1.16) we find the normal stress intensity coefficient at the points $x = \pm(b + 0)$

$$N_b = -(bk \sqrt{2b})^{-1} \left[P_2 \Phi_1(b) + \Phi_2(b)\right] \tag{2.7}$$

The solution (2.5)–(2.7) obtained can be used with confidence within the range $2 \leqslant \lambda < \infty$. For $\lambda \to \infty$ and the condition (3), from (1.12) and (2.6) we find

$$P_2\Phi_1(b) + \Phi_2(b) = -\frac{bdG}{(1-\nu)K(k)} - pb^2\left[1 - \frac{E(k)}{K(k)}\right] \qquad (2.8)$$

Equations (2.7) and (2.8) imply that when $\lambda \to \infty$, the value of N_b coincides with that obtained in the problem on splitting an elastic plane with a wedge of finite width [8]. Moreover, when $\lambda \to \infty$, the expression (2.4), with (2.2) taken into account, defining $\gamma(x)$ coincides with the analogous expression obtained in [8].

BIBLIOGRAPHY

1. Aleksandrov, V. M. and Kucherov, V. A., Certain problems on the action of two stamps on an elastic strip. Inzh. zh. MTT, №4, 1968.
2. Aleksandrov, V. M. and Vorovich, I. I., The action of a die on an elastic layer of finite thickness. PMM Vol. 24, №2, 1960.
3. Ufliand, Ia. S., Integral Transformations in the Problems of the Theory of Elasticity. L., "Nauka", 1967.
4. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products. M., Fizmatgiz, 1963.
5. Shtaerman, I. Ia., Contact Problem of the Theory of Elasticity. M., Gostekhizdat, 1949.
6. Panasiuk, V. V., Limit Equilibrium in Brittle Bodies with Cracks. Kiev, Naukova Dumka, 1968.
7. Smetanin, B. I., Some problems on slits in elastic wedges and layers. Inzh. zh. MTT, №2, 1968.
8. Markuzon, I. A., On splitting of a brittle body by a wedge of finite length. PMM Vol. 25, №2, 1961.

Translated by L. K.

CONVERGENCE OF THE PROBLEM OF LIMIT EQUILIBRIUM

PMM Vol. 34, №2, 1970, pp. 370-372

A. M. PROTSENKO
(Moscow)

(Received September 11, 1968)

The theory of limit equilibrium of a perfectly plastic body [1] usually deals with the systems possessing a finite number of degrees of freedom. The results obtained for the models of finite dimensions can be extended to the problems of limit equilibrium of solid bodies, using the methods of mathematical programing.

In the present paper we consider a perfectly plastic body of finite volume V with surface S. A load proportional to the parameter P is applied at a part of the surface denoted by S_σ. Conditions of zero displacements $u_i = 0$ ($i = 1,2,3$) (u denotes the rate of displacement vector) are given at the remainder S_u of the surface. The stress field must satisfy the equations of equilibrium and the following boundary conditions on S_σ

$$\sigma_{ij,j} = 0, \qquad \sigma_{ij}\nu_j - Pk_i = 0 \qquad (1)$$

Conditions of plasticity are assumed to have the form of a convex operator $f(\sigma) \leqslant 0$. Under these conditions the problem of limit equilibrium in static formulation consists in determination of $P^* = \sup P(\sigma)$. This corresponds to the generalized Lagrange's functional [2]